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Dualities in spin ladders

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Abstract. We introduce a set of discrete modular transformations T_ℓ, U_ℓ and S_ℓ in order to study the relationships between the different phases of the Heisenberg ladders obtained with all possible exchange coupling constants. For the two-legged ladder we show that the resonating valence bond (RVB) phase is invariant under the S_ℓ transformation, while the Haldane phase is invariant under U_ℓ. These two phases are related by T_ℓ. Moreover, there is a 'mixed' phase, that is invariant under T_ℓ, and which under U_ℓ becomes the RVB phase, while under S_ℓ becomes the Haldane phase. For odd ladders there exists only the T_ℓ transformation which, for strong coupling, maps the effective antiferromagnetic spin $\frac{1}{2}$ chain onto the spin $\frac{3}{2}$ chain. Our work is based on a combination of approximate methods such as bosonization, perturbation theory and the nonlinear sigma model, adapted to the different regimes of coupling constants.

1. Introduction

In the last two years the concept of duality has played a crucial role in understanding non-perturbative aspects of quantum field theory [1] and string theory [2]. Considering the traditional links between particle physics and statistical mechanics or condensed matter one may wonder whether these latter areas could benefit from the deeper understanding gained in the former ones. In fact, duality ideas have been important in the historical development of statistical mechanics, as shown by the Krammers–Wannier duality, order–disorder transformations, etc [3]. In this paper we shall explore the existence of duality symmetries in quantum spin systems defined on a lattice and more particularly on arrays of coupled spin chains known as spin ladders [4].

Generally speaking a duality transformation is a mapping between two models, or the same model with different parameters, which apparently have different physical properties, but which become in a way equivalent under the transformation. Dual theories usually give complementary descriptions of the same underlying phenomena.

Let us first establish what we mean by duality in a spin system. We shall consider the Heisenberg Hamiltonian defined on the d-dimensional hypercubic lattice ($d \geqslant 1$),

$$H(\{J_\mu\}) = \sum_{\boldsymbol{\mu}} \sum_{\boldsymbol{x}} J_\mu \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+\mu} \tag{1}$$

where $\boldsymbol{S}_x$ is a spin S matrix acting at the position $\boldsymbol{x} = (x_1, \ldots, x_d)$, and $\boldsymbol{\mu}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{\mu}_d = (0, 0, \ldots, 1)$.

We shall define the dual of the Hamiltonian (1) as a Hamiltonian $H_D = H(\{J^D_{\boldsymbol{\mu}}\})$ characterized by a new set of coupling constants $\{J^D_{\boldsymbol{\mu}}\}$, and such that the low-energy

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spectrum of H and H_D is in one-to-one correspondence. This implies that the free energy and the ground-state (g.s.) energy will also be the same for both models.

In the classical limit where the spin $S \gg 1$, the g.s. of (1) is given by the classical minima,

$$\boldsymbol{S}_{\boldsymbol{x}} = S\boldsymbol{n} \prod_{\boldsymbol{\mu}} \epsilon_{\boldsymbol{\mu}}^{\boldsymbol{x}\cdot\boldsymbol{\mu}} \tag{2}$$

where $\boldsymbol{n}$ is a 3-component unit vector and $\epsilon_{\boldsymbol{\mu}} = -\mathrm{sign}J_{\mu}$. The energy of (2) is given by,

$$E_0^{\mathrm{class}} = -S^2 \sum_{\boldsymbol{\mu}} \sum_{\boldsymbol{x}} |J_{\boldsymbol{\mu}}|. \tag{3}$$

The signs of the exchange coupling constants, ϵ_{μ}, determine the type of order parameter which characterizes the g.s. Thus $\epsilon_{\mu} = 1$ or -1 correspond to ferromagnetic (F) or antiferromagnetic (A) order in the direction μ of the lattice. Altogether there are 2^d possible classical vacua which we denote, for $d = 1$ and 2, by the sequences,

$$\begin{aligned} &d = 1 : A, F \\ &d = 2 : AA, AF, FA, FF. \end{aligned} \tag{4}$$

The energies of the classical g.s. and the excitations are independent of the type of vacua (4). All the classical Heisenberg models are equivalent. However the quantum corrections drive them into very different quantum vacua. Only the pure ferromagnetic system (i.e. $J_{\mu} < 0, \forall\mu$), survives the quantum fluctuations, but the non-ferromagnetic systems deeply change their structure. The purpose of this paper is to show the relations existing between the different vacua by means of a certain type of duality transformation.

At this point it is useful to make an analogy between two-dimensional (2D) spin systems and fermions living on a 2D torus [5]. To define a fermion on a torus one has to specify the boundary conditions along the a and b cycles. They can be periodic (P) or antiperiodic (A). This gives rise to four possible spin structures, labelled as AA, AP, PA and PP, which mix under the action of the modular transformations T, U and S as follows [5],

$$\begin{aligned} &T : AA \leftrightarrow AP, PA \leftrightarrow PA, PP \leftrightarrow PP \\ &U : AA \leftrightarrow PA, AP \leftrightarrow AP, PP \leftrightarrow PP \\ &S : AP \leftrightarrow PA, AA \leftrightarrow AA, PP \leftrightarrow PP. \end{aligned} \tag{5}$$

Observe that the spin structure PP is left invariant under the action of the modular group. The fermion determinant with the boundary conditions AA, AP, PA turns out to be given by Jacobi ϑ functions, which transform among themselves under the modular group as described by (5). The fermion determinant for PP boundary conditions is zero due to the existence of a zero mode.

In the case of 2D spin systems the role of the cycles a and b is played by the directions $\boldsymbol{\mu}_1 = (1, 0)$ and $\boldsymbol{\mu}_2 = (0, 1)$. The analogue of the spin structure is given by the (anti)ferromagnetic nature of the coupling $J_{\boldsymbol{\mu}}$ along the directions $\boldsymbol{\mu}_{1,2}$. Finally, a modular transformation is a redefinition of the unit cell of the lattice. In the case of spin ladders the above analogies can be collected in the following dictionary,

$$\begin{aligned} &\text{Torus Lattice} \leftrightarrow \text{Spin Ladder} \\ &a - \text{cycle} \leftrightarrow \text{legs} \\ &b - \text{cycle} \leftrightarrow \text{rungs} \\ &\text{Antiperiodic boundary condition (BC)} \leftrightarrow \text{Antiferromagnetic Coupling} \\ &\text{Periodic BC} \leftrightarrow \text{Ferromagnetic Coupling} \\ &\text{Modular Transformation} \leftrightarrow \text{Bond Moving Transformation} \end{aligned} \tag{6}$$

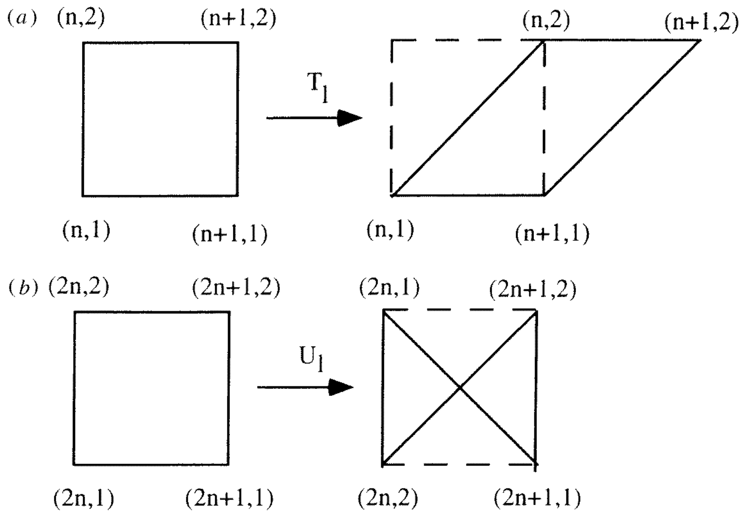

Figure 1. (*a*) Action of the transformation T_ℓ on the 2-leg ladder. (*b*) Action of the transformation U_ℓ on the 2-leg ladder.

These correspondences have an analogue in $d > 2$. On a two-legged ladder (2-ladder), we shall define three transformations T_ℓ, U_ℓ and S_ℓ as follows. The T_ℓ transformation consists of the shift by one lattice spacing of one leg with respect to the other (see figure 1(*a*)). The U_ℓ transformation consists of the permutation of the two sites of the even rungs, while leaving invariant the odd ones (see figure 1(*b*)). Finally the S_ℓ transformation is defined by the equation $S_\ell = T_\ell U_\ell T_\ell$, and has the effect of converting all the vertical bonds (rungs) into horizontal ones (legs), while half of the horizontal bonds become vertical bonds and the other half become diagonal bonds of length $\sqrt{5}$. We remark that T_ℓ, U_ℓ and S_ℓ do not generate the standard modular group.

Using these definitions one can see that the classical vacua AA, AF, FA and FF, get mixed under the action of T_ℓ, U_ℓ, S_ℓ in the form described by (5), with the replacement: (anti)periodic $\leftrightarrow$ (anti)ferromagnetic. The term 'bond moving' in (6) refers to a transformation introduced by Migdal and Kadanoff in the study of the Ising model with renormalization group methods [6]. As a fermion on a lattice with non-periodic BCs can be regarded as essentially the same object, we conjecture that the 2-ladder, and more generally the ladders with an even number of legs, with 'magnetic structures', AA, AF, FA, belong to the same universality class and are related through dual transformations. This conjecture implies in particular to the equivalence between the RVB state (AA couplings) and the Haldane state (AF couplings) of the 2-ladder studied by various authors [7, 8], but it also suggests new equivalences which have not been studied so far involving a 'mixed' state corresponding to the FA couplings. The use of the term duality applied to ladders may lead to the erroneous conclusion that the magnetic structures AA, AF and FA yield different phases separated by well-defined phase boundaries. This is not the case for all the states AA, AF and FA belong indeed to the same quantum phase. The role of the duality transformations is to show the equivalence between different ladder's states from a perspective closer in spirit to what is called nowadays duality in the realm of particle physics, and which we illustrated above in the example of modular transformation of spin structures in a torus.

We shall confine ourselves in this paper to the case of the spin $\frac{1}{2}$ ladder with two legs,

Table 1.

	T_ℓ	U_ℓ	S_ℓ
$\|a\rangle$	$-\|a\rangle$	$\|a\rangle - \|b\rangle$	$\|b\rangle$
$\|b\rangle$	$-\|a\rangle + \|b\rangle$	$-\|b\rangle$	$\|a\rangle$

trying to prove the above conjecture using perturbative and field theoretical techniques. At the end we shall briefly consider the case of odd ladders.

Let us start with a toy ladder.

2. A 2 × 2 cluster

The simplest 2-ladder has four spins coupled by the Hamiltonian,

$$H = J_a(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_3 \cdot \boldsymbol{S}_4) + J_b(\boldsymbol{S}_1 \cdot \boldsymbol{S}_4 + \boldsymbol{S}_3 \cdot \boldsymbol{S}_2). \tag{7}$$

The g.s. of the non-ferromagnetic Hamiltonians (i.e. $\epsilon_{a,b} \neq 1$) is a singlet and therefore can be written as the linear combination

$$\begin{aligned} &|\psi\rangle = \tau|a\rangle + |b\rangle \\ &|a\rangle = (12)(34) \qquad |b\rangle = (14)(32) \end{aligned} \tag{8}$$

where (ij) denotes the singlet valence-bond state constructed from the spin $\frac{1}{2}$'s located at the sites i and j. The transformations S_ℓ, T_ℓ and U_ℓ of figures 1(a) and (b) become, for the toy ladder, elementary transpositions,

$$T_\ell : 3 \leftrightarrow 4, U_\ell : 2 \leftrightarrow 3, S_\ell : 2 \leftrightarrow 4. \tag{9}$$

The action of T_ℓ, U_ℓ, S_ℓ on the states (8) can be easily derived from (8) and (9). They are given in table 1.

The 'modular transformations' induced on the 'modular parameter' τ that follow from table 1 are,

$$T_\ell : \tau \rightarrow -(\tau + 1), U_\ell : \tau \rightarrow -\tau/(\tau + 1), S_\ell : \tau \rightarrow 1/\tau \tag{10}$$

which are similar but not identical to the standard modular transformations of the torus.

The g.s. energy of (7) is,

$$E = -\tfrac{1}{2}(J_a + J_b) - \sqrt{J_a^2 + J_b^2 - J_a J_b} \tag{11}$$

corresponding to a value of τ given by,

$$\tau = -1 + \frac{J_a}{J_b} - \epsilon_b \sqrt{1 - \frac{J_a}{J_b} + \left(\frac{J_a}{J_b}\right)^2}. \tag{12}$$

The values of τ obtained by changing the signs and strengths of the exchange coupling constants $J_{a,b}$, cover the whole real axis as described in table 2.

Table 2.

(ϵ_a, ϵ_b)	AF	FF	FA	AA
τ	$(-\infty, -2)$	$(-2, -\frac{1}{2})$	$(-\frac{1}{2}, 0)$	$(0, \infty)$

Table 3.

(J_a, J_b)	State	τ	S_ℓ	U_ℓ	T_ℓ
AA	RVB	1	RVB	MIX	HAL
AF	HAL	-2	MIX	HAL	RVB
FA	MIX	$-\frac{1}{2}$	HAL	RVB	MIX

We have included the case FF which corresponds to an excited state, since the g.s. is a spin 2 multiplet. The S_ℓ transformation (10) leaves invariant the AA and FF domains, while interchanges the regions AF and FA. S_ℓ duality is an exact symmetry of the Hamiltonian (7). Actually, $\tau = 1$ is a fixed point of S_ℓ. The T_ℓ and U_ℓ transformations are approximate symmetries in the sense that the Hamiltonian (7) is not mapped into a similar one with a redefinition of $J_{a,b}$. However, one can see that $\tau = -\frac{1}{2}$ is a fixed point of T_ℓ, while the AA region $\tau > 1$ is mapped under T_ℓ into the AF region $\tau < -2$. Similarly $\tau = -2$ is a fixed point of U_ℓ, while the AA region $0 < \tau < 1$ is mapped under U_ℓ into the FA region $-\frac{1}{2} < \tau < 0$. All this shows that equations (5) hold with some caveats for the 2×2 cluster.

Within each domain, AA, AF and FA, we shall choose a representative state $|\tau\rangle$ with the property of being invariant under one of the dual transformations. The state $\tau = 1$ can be called a RVB state since it describes the resonance between two vertical and horizontal bonds. The state $\tau = -2$ is a Haldane-like state (HAL) in the sense that it is obtained upon forming the spin 1 state along the rungs, which then couple to form a singlet. Finally $\tau = -\frac{1}{2}$ is a mixed state (MIX), corresponding to ferromagnetic chains coupled antiferromagnetically. Moreover, each of the states $|RVB\rangle$, $|HAL\rangle$ and $|MIX\rangle$ gets transformed into another by the action of T_ℓ, U_ℓ, S_ℓ.

The results are summarized in table 3.

For 2-ladders with a large number of rungs we can still make sense of the transformation properties collected in table 3. In that case $|RVB\rangle$ denotes the g.s. of a ladder with AA couplings, etc.

The rest of the paper will be devoted to showing the validity of table 3.

3. The weak-coupling regime: T_ℓ duality

If the two legs are weakly coupled (i.e. $|J_a/J_b| \gg 1$), the T_ℓ duality becomes a manifest symmetry of the effective low-energy theory.

For $J_a > 0$ we can use bosonization techniques to show this fact. Indeed the effective ladder Hamiltonian can be written in the bosonized model as [10],

$$\begin{aligned} H = H_{\mathrm{WZW}} &+ \lambda_1(\boldsymbol{J}_L \cdot \boldsymbol{J}_R + \hat{\boldsymbol{J}}_L \cdot \hat{\boldsymbol{J}}_R) + \lambda_2(\boldsymbol{J}_L \cdot \hat{\boldsymbol{J}}_R + \hat{\boldsymbol{J}}_L \cdot \boldsymbol{J}_R) \\ &+ \lambda_3 \operatorname{Tr}(g\boldsymbol{\sigma}) \operatorname{Tr}(\hat{g}\boldsymbol{\sigma}) + \lambda_4 \operatorname{Tr} g \operatorname{Tr} \hat{g} \end{aligned} \tag{13}$$

where g and $\boldsymbol{J}$ (resp. $\hat{g}$ and $\hat{\boldsymbol{J}}$) are the WZW field and current which bosonize the upper (lower) spin chains of the ladder. The initial values of the different coupling constants appearing in (13) are given by

$$\lambda_1 < 0 \qquad \lambda_2 = \lambda_3 = J_b \qquad \lambda_4 = 0. \tag{14}$$

In the bosonized representation the translation of a chain by one site is equivalent to the discrete symmetry [9], $g \to -g$. Therefore the operator T_ℓ is realized in the WZW model by the map,

$$T_\ell : (g, \hat{g}) \rightarrow (-g, \hat{g}) \tag{15}$$

implying that T_ℓ is equivalent to the following change of couplings,

$$\begin{aligned} J_a &\xrightarrow{T_\ell} J_a \\ J_b &\xrightarrow{T_\ell} -J_b. \end{aligned} \tag{16}$$

Equation (16) illustrates the relations $T_\ell|RVB\rangle = |HAL\rangle$ and $T_\ell|HAL\rangle = |RVB\rangle$, which establish the equivalence between the RVB and Haldane states in the weak coupling limit. White has observed [8] that the spins located in diagonal positions of the 2-ladder tend to form effective spins 1, and using, what we call the T_ℓ transformation, he shows the equivalence between the two phases. This is done in [8] by introducing diagonal couplings in order to connect continously the $|RVB\rangle$ and $|HAL\rangle$ states. What we show in this paper is that this connection can be also thought of as a discrete modular transformation by which the properties of both models can be put in one-to-one correspondence.

If $J_a < 0$ both legs are in a ferromagnetic state with total spin $S_{\rm tot} = N/2$. A weak antiferromagnetic coupling, $J_b > 0$, splits this degeneracy giving a state which, to first order in perturbation theory, is given by the singlet appearing in the Clebsch–Gordan decomposition $S_{\rm tot} \times S_{\rm tot}$. Obviously, the latter state is invariant under a shift of one of the legs. Thus the state $|MIX\rangle$, in the weak coupling regime, is invariant under T_ℓ duality, according to table 3.

4. The strong coupling regime: U_ℓ duality

In the strong coupling regime (i.e. $|J_a/J_b| \ll 1$) the rung Hamiltonian yields the zero-order approximation, while the leg Hamiltonian acts as a perturbation.

The rungs, in an AF ladder, are mostly in the spin 1 state which couple antiferromagnetically along the leg direction, yielding effectively a Haldane chain. The U_ℓ transformation, which simply permutes the two spins on the even rungs, leaves invariant the corresponding Haldane state (i.e. $U_\ell|HAL\rangle = |HAL\rangle$).

Next we shall show using perturbation theory that the RVB and MIX states are exchanged by U_ℓ duality.

For $J_b >> 1$ the rungs are in a singlet state. The g.s. energy computed to second order in J_a is given by [11],

$$E_0/N = -\frac{3}{4}J_b - \frac{3}{8}\frac{J_a^2}{J_b}. \tag{17}$$

The first excited states form a band of spin 1 magnons,

$$|k\rangle = \frac{1}{\sqrt{N}}\sum_{x=1}^{N} \mathrm{e}^{\mathrm{i}kx}|x\rangle \tag{18}$$

where $|x\rangle$ denotes the state with singlets on all rungs except at the position x where it is a triplet. The dispersion relation $\omega(k)$ of (18) is given, to second order in J_a, by [11],

$$\omega(k) = J_b + J_a \cos k - \frac{1}{4}\frac{J_a^2}{J_b}(3 - \cos 2k). \tag{19}$$

The action of U_ℓ on the magnons (18) is,

$$U_\ell|k\rangle = |k + \pi\rangle. \tag{20}$$

Table 4. Parameters of the 2-ladder with spin S.

	AA	AF	FA
θ	0	$4\pi S$	0
g	$\frac{1}{S}\left(1+\frac{J_b}{2J_a}\right)^{1/2}$	$\frac{1}{S}$	$\frac{1}{S}\left(\frac{J_b}{2\lvert J_a\rvert}\right)^{1/2}$

Hence the spectrum of the RVB and MIX states, up to second order in J_a, are exchanged under U_ℓ duality, as can be seen from the following identities satisfied by (17) and (19),

$$\begin{aligned} &E_0(J_a, J_b) = E_0(-J_a, J_b) \\ &\omega(k, J_a, J_b) = \omega(k+\pi, -J_a, J_b). \end{aligned} \tag{21}$$

5. The intermediate coupling regime: S_ℓ duality

When $|J_a/J_b| \sim 1$, the effective theory can be obtained by mapping the ladder into the nonlinear sigma model (NLSM) [12–14]. The values of the NLSM coupling constants are given in table 4,

From these equations we obtain the curious relation,

$$g_{AA}^2 = g_{AF}^2 + g_{FA}^2. \tag{22}$$

S_ℓ duality corresponds to the permutation of vertical and horizontal bonds. Since on a 2-ladder there are twice as many horizontal bonds than vertical ones we expect a perfect balance between both couplings whenever $2|J_a| = |J_b|$. In this case $g_{AF} = g_{FA} = g_{AA}/\sqrt{2}$. The change of θ by $4\pi S$, when going from FA to AF, does not affect the physics of the problem and recalls what happens with duality transformation in field theories [1].

Extrapolating the NLSM map away the intermediate couplings we still find an agreement with table 3. In the strong-coupling regime both g_{AA} and g_{FA} go to the same asymptotical value, which agrees with the fact that the U_ℓ operation maps one g.s. into the other. On the other hand, the value $g_{AF} = 1/S = 2$ corresponds to the NSLM coupling of a spin chain with spin 1.

In the weak coupling regime the NLSM map is not reliable, however, we see from table 4, that in that limit $g_{AA} = g_{AF} \gg g_{FA}$ which agrees with the fact that T_ℓ interchages the RVB and HAL states. Of course in this limit we have two weakly coupled chains, which should be treated with bosonization techniques.

Summarizing our results we can say that the Haldane and the mixed phases are S_ℓ-dual, while the RVB phase is self-dual under an S_ℓ transformation.

6. Bond-moving dualities

What is the origin of the duality properties of ladders? In conformal field theory or string theory duality is an expression of modular invariance. Something of this sort exists also for spin systems. To show this we shall use a generalization of the Migdal–Kadanoff transformations, which consist of the substitution of couplings between nearest-neighbour sites by other nearest-neighbour or next-nearest-neighbour couplings [6]. This is achieved by adding a potential V to the Hamiltonian H, so that the new Hamiltonian $H' = H + V$ has a g.s. energy (and free energy) E' smaller than the g.s. energy E of H, provided

$\langle V \rangle = 0$, where the vacuum expectation value is taken with respect to the g.s. of H [6]. From figure 1(a)) we observe that the T_ℓ-transformation corresponds to the bond moving potential,

$$V_{T_\ell} = \sum_n (J_b' \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_2(n+1) - J_b \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_2(n)). \tag{23}$$

The Hamiltonian obtained by adding (23) to the ladder Hamiltonian $H(J_a, J_b)$, is a new Hamiltonian $H(J_a', J_b')$, where $J_{a,b}$ are given by

$$\begin{aligned} J_a' &= J_a \\ J_b' &= J_b \langle \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_2(n) \rangle / \langle \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_2(n+1) \rangle. \end{aligned} \tag{24}$$

The expectation values in (24) are computed with respect to the g.s. of $H(J_a, J_b)$. Equation (24) implies $\mathrm{sign}(J_b') = -\epsilon_a \epsilon_b$, which indeed corresponds to a T_ℓ transformation.

Similarly the bond-moving transformation which corresponds to U_ℓ gives,

$$\begin{aligned} J_b' &= J_b \\ J_a' &= J_a \langle \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_1(n+1) \rangle / \langle \boldsymbol{S}_1(n) \cdot \boldsymbol{S}_2(n+1) \rangle. \end{aligned} \tag{25}$$

Finally the S_ℓ-tranformation can be derived from its definition $S_\ell = T_\ell U_\ell T_\ell$. For all these transformations, duality would amount to the equality $E(J_a, J_b) = E(J'_a, J'_b)$. The variational principle underlying the Migdal–Kadanoff transformation only guarantees that $E(J_a, J_b) > E(J_a', J_b')$, however, after the results obtained above using perturbative and field theoretical methods, we have good reasons to believe that the replacement of inequalities by equalities for the energies and free energies yields a good approximation. Further studies are necessary to fully establish these facts.

7. Beyond the 2-ladder

Most of the results shown so far are generalizable to the case of even ladders with periodic BCs along the rungs. The T_ℓ transformation is given by the shift of one lattice space of the even legs with respect to the odd legs, so that the rungs become zigzag lines across the ladder. In fact, this definition applies to all types of ladders, even and odd, with different BCs across the rungs. The U_ℓ transformation consists of the shift by one lattice space of the even rungs, so that the legs become zigzag lines along the ladder. The new coupling constants, obtained upon these transformations, are also given by equations (24) and (25).

For odd ladders with open BCs along the rungs, there does not seem to be a sensible definition of the U_ℓ and S_ℓ transformations, as we did above for T_ℓ. The odd ladders of type AA are in the same universality class as the spin $\frac{1}{2}$ antiferromagnetic Heisenberg chain, whose g.s. we shall denote as $|A_{1/2}\rangle$. On the other hand the n_ℓ ladders (n_ℓ: odd) with AF couplings are, at least in the strong-coupling regime, equivalent to spin $n_l/2$ antiferromagnetic chain, whose g.s. we shall denote as $|A_{n_\ell/2}\rangle$. The role of T_ℓ is to exchange the AA and AF couplings, which implies that $T_\ell|A_{1/2}\rangle = |A_{n_\ell/2}\rangle$. The previous equivalence can be established for every regime of couplings using, as we did for the 2-ladder, the appropriate technique. Thus for intermediate couplings, where we can use the mapping of the ladder into the NLSM [13], we obtain the parameters $\theta_{AA} = \pi$ and $\theta_{AF} = \pi n_\ell$, which coincide modulo 2π. The odd ladders with FA and FF couplings are equivalent to ferromagnetic Heisenberg chains with spins $\frac{1}{2}$ and $n_\ell/2$ respectively. Their g.s. is invariant under T_ℓ.

In summary we have seen that the even and odd ladders have quite different duality properties which is of course a manifestation of the fact that they both belong to different universality classes.

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